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Models of intermediate spectral statistics

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Based on numerical results it is conjectured that the spectral statistics of certain pseudointegrable billiards have a special form similar to that of the Anderson model at the transition point. A simple theoretical model where such statistics can be obtained analytically is briefly discussed. A few models with similar behavior are considered. In particular, we analytically found the eigenvalue statistics of a Poisson-distributed matrix perturbed by a rank one matrix, which is a good model for spectral statistics of a singular billiard.
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A major area of interest in the theory of quantum chaos is the relation between the classical dynamics of a system and the spectral statistics of its quantum counterpart. Integrable dynamics generally leads to uncorrelated energy levels (Poisson statistics) and completely chaotic dynamics corresponds to the Wigner statistics of one of the standard random matrix ensembles (RME). There is a lot of numerical evidence (see, e.g., [1]) and some analytical arguments [2] in favor of these conjectures.

These statistics are also supposed to be valid in the limiting regimes [far from the metal-insulator transition (MIT) point] of the Anderson model in three dimensions (see, e.g., [3]), while there are strong indications that the level statistics exactly at the MIT point constitute a third universal ensemble [4] (connected with fractal nature of wave functions [5]) whose main features are (i) the existence of level repulsion (as in RME), and (ii) slow (approximately exponential) fall-off of the nearest-neighbor distribution at large distances [the opposite of $\exp(-s^2)$ behavior in RME].

The purpose of this Rapid Communication is to point out that a similar phenomenon also exists for certain dynamical systems. First we present numerical results that demonstrate that the spectral statistics of pseudointegrable billiards have many similiarities with the critical statistics of the Anderson model in the MIT point. We also observe that statistical properties of a few pseudointegrable billiards are quite well described by a simple theoretical model of intermediate statistics which we called the semi-Poisson statistics. A few other dynamical models (the Kepler billiard and the rough billiard) where the analogous behavior has been observed are shortly discussed. Finally, we consider two models (of independent interest) where the existence of intermediate statistics can be proved analytically.

The above-mentioned conjectures about spectral statistics are applied only to the limiting cases of classical dynamics, namely completely integrable or completely chaotic ones. But there are models which are neither integrable nor chaotic. The simplest example is given by plane polygonal billiards with all angles equal rational multiples of π (called pseudointegrable systems) where all trajectories belong to a surface of genus larger than one (see [6] and references therein).

We have performed extensive numerical calculations for plane billiards in the shape of right triangles with one angle equals π/n and Dirichlet boundary conditions. For all $n = 5,7,\ldots,30$ typically 20 000 energy levels have been computed with a precision better than 10^{-2} of the mean level density. In Fig. 1 we present the cumulative spacing distribution (integral of nearest-neighbor distribution) for low-energy part of the spectrum for the billiard with $n=5$. The cumulative spacing distributions for Poisson statistics $N_P(s)=1-\exp(-s)$ (dotted line), for the Wigner surmise $N_W(s)=1-\exp(-\pi s^2/4)$ (dashed line), and for a new statis-

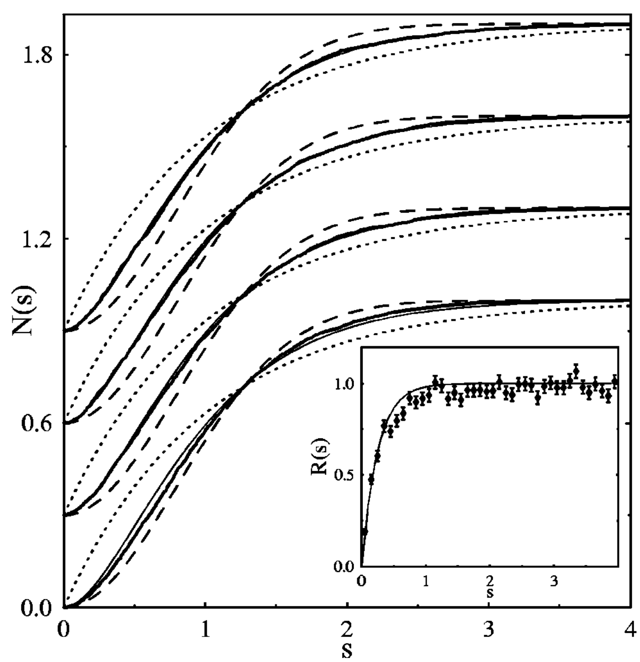

FIG. 1. Cumulative spacing distribution for four slices of 2500 consecutive energy levels for a right triangular billiard with $\pi/5$ angle (energy increasing from bottom to top). Inset: the two-point correlation function from all 10 000 levels.

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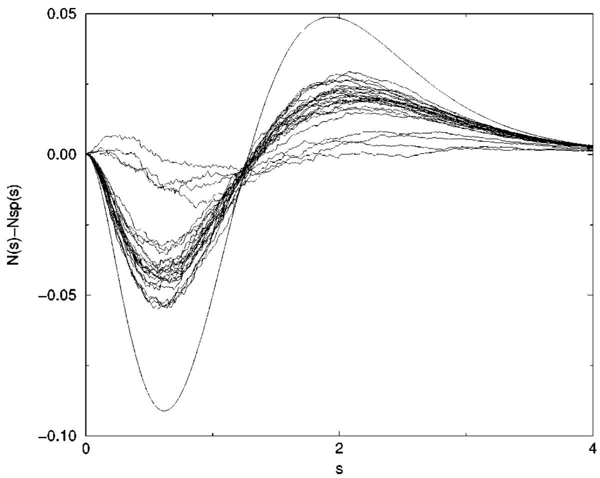

FIG. 2. Difference between the cumulative spacing distribution for triangular billiards with $n=5,7,\ldots,30$ and the semi-Poisson one. Solid line, this quantity for the GOE distribution. For details, see text.

tics which we call the semi-Poisson statistics (solid line) [7],

$$p_{sp}(s)=4se^{-2s}, \quad N_{sp}(s)=1-(2s+1)e^{-2s} \tag{1}$$

are also plotted for each slice.

The semi-Poisson statistics are a particular case of the short-range plasma model (SRPM) with interaction only between closest neighbors discussed later where all correlation functions can be computed in the closed form. In the inset of Fig. 1 we show the two-point correlation function computed from the first 10 000 levels of the above triangular billiard together with the analytical result for this quantity for the semi-Poisson statistics

$$R_2(s)=1-e^{-4s}. \tag{2}$$

Figure 1 indicates a transition from a spacing distribution close to the Wigner surmise for the bottom of the spectrum to a distribution which (i) seems to be stable with respect to increasing energy, and (ii) is close to the semi-Poisson distribution given by Eq. (1).

In Fig. 2 we plot the difference between the cumulative spacing distribution for all 20 000 levels except the first 5000 for triangular billiards with different n and the semi-Poisson distribution (1). On analyzing these and other results we come to the following conclusions:

(i) Spectral statistics of the above billiards depend (in the first approximation) on one parameter q, which we propose to call arithmetical genus. For odd n it equals the usual geometrical genus g of a surface associated with classical motion on pseudointegrable billiards [6] but for even n, $q=\phi(n)/2$ where $\phi(n)$ is the Euler function. The four visible groups of lines in Fig. 2 (from the top to the bottom) correspond respectively to triangles with $q=2$ $(n=5,8,10,12)$, with $q=3$ $(n=7,14,18)$, with $q=4$ $(n=9,16,20,24,30)$, and all the rest.

(ii) The spectral statistics for triangles with $q=2$ $(n=5,8,10,12)$ are quite well described (better than 10^{-2}) by the semi-Poisson distribution (1). We have checked that even the next-to-nearest distribution for these cases is close to the one predicted in SRPM with closest neighbors interaction (the semi-Poisson statistics)

$$p(2,s)=\frac{8}{3}s^3\exp(-2x). \tag{3}$$

(iii) Numerical data suggest that with increasing q spectral statistics tends to a certain limiting distribution that is close to the one of SRPM with three nearest-neighbors interaction.

We stress that (a) these statements are based mainly on numerical results and at present they should be considered as conjectures, (b) triangular billiards in the shape of right triangles with angle π/n belong to the so-called Veech polygons [8], which have an interesting mathematical structure and may not be generic.

To introduce SRPM (which serves us as the simplest analytical model of intermediate statistics) we remind one that the joint probability distribution of eigenvalues of RME can be written as the probability distribution of a one-dimensional gas of classical particles interacting pairwise through the repulsive potential $V(x)=-\log|x|$,

$$P_N(x_1,\ldots,x_N)=Z_N^{-1}\exp\left(-\beta\sum_{i<j}V(x_j-x_i)\right), \tag{4}$$

where Z_N is the partition function and the inverse temperature is fixed to $\beta=1$, 2, and 4 for orthogonal, unitary, and symplectic ensembles, respectively.

Now let us consider $N+2$ particles with positions x_j in an interval of size L and take $0=x_0<x_1<\cdots<x_N<x_{N+1}=L$. We choose their joint probability distribution as in Eq. (4), but with the interaction restricted only to a finite number of nearest-neighbor particles. Instead of the sum over all $i<j$ we shall sum only over particles with $0<j-i\leqslant h$, where h is the number of interacting neighbors. In some sense this model belongs to a class of models with ''screened'' two-body potential similar, e.g., to the Gaudin model [9]. An important property of this model is that its statistical properties can be computed analytically for any form of two-body potential $V(x)$ and any h [10,11]. For the natural choice $V(x)=-\log|x|$ and $h=1$ (it is this model we called the semi-Poisson statistics) one finds that in the large N limit the distribution of n nearest neighbors is

$$P(n,s)=\frac{(\beta+1)^{n(\beta+1)}}{\Gamma(n(\beta+1))}s^{n(\beta+1)-1}e^{-(\beta+1)s}, \tag{5}$$

which for $\beta=1$ and $n=1,2$ give Eqs. (1) and (3). The two-point correlation function is given by Eq. (2) and the number variance by

$$\Sigma^2(L)=\frac{L}{2}+\frac{1}{8}(1-e^{-4L}). \tag{6}$$

The model with $-\log|x|$ interaction between nearest neighbors is extremely simple, has no adjustable parameters (except h), and (i) it exhibits level repulsion as for standard random matrix ensembles: $p(s)\to s^\beta$ as $s\to 0$, (ii) $p(s)$ decreases as $\exp(-\Lambda s)$ for large s with $\Lambda=\beta h+1$, (iii)

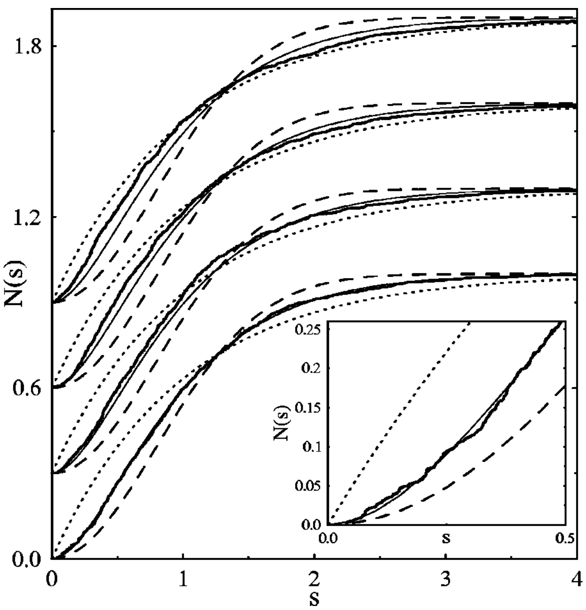

FIG. 3. Cumulative nearest-neighbor distribution for four slices of successive 1000 energy levels for the Kepler billiard. Inset: small-s behavior for the first slice.

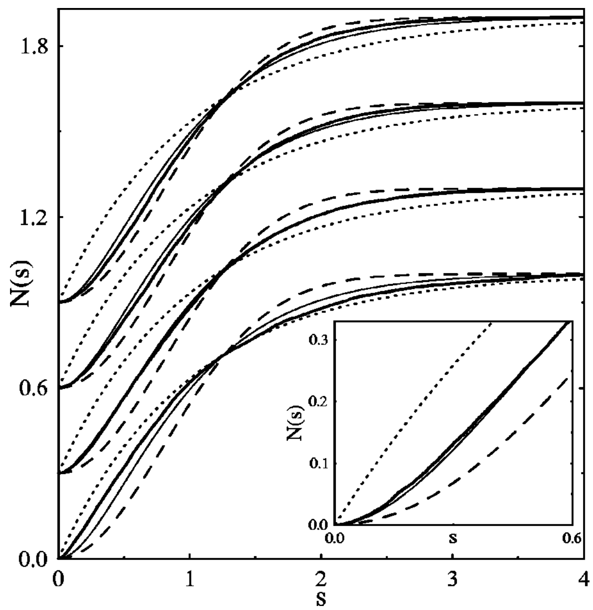

FIG. 4. Same as in Fig. 3 but for the rough billiard. Inset: small-s behavior for the second slice.

$\Sigma^2(L)\to\chi L$ when $L\to\infty$ with $\chi=1/\Lambda$ [23]. It is this type of feature that one requires for the critical statistics of the Anderson model at the MIT point (see [12] and references therein).

We found several other dynamical models where intermediate spectral statistics well described by the above semi-Poisson distribution have been observed. In these cases the intermediate statistics are not the limiting distribution, as seems to be the case for pseudointegrable billiards, but rather a transient phenomenon. The first model (which we call the Kepler billiard) was proposed in [13] and consists of a rectangular billiard with a Yukawa-type potential,

$$V(r)=\lambda\frac{e^{-\kappa r}}{r}, \tag{7}$$

where r is the distance from a certain point inside the rectangle (usually in the center).

We have numerically computed 4000 energy levels of the Kepler billiard with sides $a=4$ and $b=\pi$ imposing the boundary conditions $\psi(x+a,y)=\exp(i\phi_1)\psi(x,y)$ and $\psi(x,y+b)=\exp(i\phi_2)\psi(x,y)$ with $\phi_1=(\sqrt{5}-1)\pi/2$ and $\phi_2=1/4$. The result for $\lambda=8$, $\kappa=8$ is presented in Fig. 3. Though with increasing energy the level statistic of this billiard has to tend to the Poisson distribution [13], for low energies the nearest-neighbor distribution is quite close to the semi-Poisson distribution in Eq. (1).

We have also considered the rough billiard discussed in [14], which is a small deformation of a circular billiard. In polar coordinates its boundary is defined by

$$r(\theta)=r_0\left(1+\sum_{n=2}^{N}\left[a_n\cos(n\theta)+b_n\sin(n\theta)\right]\right), \tag{8}$$

where r_0, a_n, and b_n are constants. We chose a_n and b_n at random with uniform distribution in the interval $[-c,c]$. The value of c has been fixed by the requirement that $|r(\theta)-r_0|/r_0\leqslant 0.05$ for all θ and then r_0 has been calculated from the condition that the surface of the resulting figure equals 4π.

In Fig. 4 we present the result of a numerical calculation of the first 500 energy levels divided into four slices, 1-200, 100-300, 200-400 and 300-500, averaged over 40 configurations (in total, 20 000 eigenvalues). In this case one observes a transition from the Poisson distribution to the GOE distribution (as it should be), but in between there is an interval of energy where the resulting distribution is quite close to the semi-Poisson one (1). Intermediate statistics quite well described by Eqs. (1) and (2) were also observed in nonhydrogenic atoms in external fields [15] and in the three-dimensional Anderson model with special boundary conditions at the transition point [16].

The above collection of numerical results suggests that a new type of spacing distribution, which in the simplest cases is well approximated by Eq. (1), has a general significance. The characteristic feature of this distribution is a linear level repulsion and exponential falloff at large spacings. Though the degree of universality of this distribution is not yet clear it is found for many systems.

The common origin of this kind of intermediate statistics appears to be connected with the (multi)fractal character of typical wave functions either in the coordinate space (as with the Anderson model [5]) or in the reciprocal space (as it seems to be the case for above-mentioned dynamical systems). The complete theory of such intermediate statistics is still to be developed. Though there exist a few standard models of crossover between the Poisson and RME statistics [17–19], we have checked numerically that none of them is capable of detailed description of intermediate statistics discussed in this paper. The main reason for it is related to the results of Ref. [20], where it was proved that fractal properties of wave functions quite naturally lead to (a) small correlations between levels with different energies (simply because these wave functions ''live'' on different fractals) and (b) strong correlations between levels with nearby energies.

This peculiar behavior of wave function overlap is completely absent in the above-mentioned models but is inherent in plasma models with screened Coulomb interaction (like SRPM discussed above or the Gaudin model [9]), which we think are adequate for analytical description of intermediate statistics.

Another class of models that has analogous features (but without fractal properties and with $\Sigma^2/L \to 1$) can be constructed as follows. Let us consider an $N\times N$ matrix of the form $H_{mn}=e_n\delta_{mn}+t_m t_n$, where e_n are mutually independent random variables uniformly distributed between $-W$ and W and t_m is a fixed vector. Eigenvalues E of this matrix obey the equation $\Sigma_{n=1}^{N} r_n/(E-e_n)=1$, where $r_n=|t_n|^2$. The density of these eigenvalues can be transformed to the following form:

$$\rho(E)=\int \frac{d\alpha}{2\pi} e^{-i\alpha} \sum_{n=1}^{N} \frac{r_n}{(E-e_n)^2} \prod_{m=1}^{N} \exp\left(i\alpha \frac{r_m}{E-e_m}\right).$$

As each e_n is an independent random variable this expression permits the explicit computation of the correlation functions $R_k(E_1,\ldots,E_k)=\langle \rho(E_1)\cdots\rho(E_k)\rangle$, where $\langle\cdots\rangle$ denotes the mean value over all e_n. We state here only the behavior of the two-point correlation function when $r_n=r$ in the limit $N\to\infty$ at small $\epsilon=E_2-E_1$, $R_2(\epsilon)\to(\pi\sqrt{3}/2)\epsilon\approx 2.72\epsilon$, which is different from the GOE prediction $R_2^{\mathrm{GOE}}(\epsilon)\to(\pi^2/6)\epsilon\approx 1.64\epsilon$.

The matrix model discussed above is closely related to a singular billiard proposed in [21]. The same method can also be applied to the Bohr-Mottelson model [22].

To summarize, we have demonstrated numerically that a level spacing distribution, which is well approximated by Eq. (1), occurs for quite different dynamical systems (at least in certain energy ranges), and thus seems to have a general character (connected probably with the fractal nature of eigenfunctions [5,12]). This conjecture is supported by the analogy between dynamical and disordered systems. We have also developed a one-dimensional gas model with nearest-neighbor interaction that leads to Eqs. (1) and (2). Finally, we have demonstrated analytically that certain matrix models also lead to similar statistics.

Note added. When this paper had been completed we became aware that A. Pandey [23] had investigated the model equivalent to our SRPM in the framework of band random matrix theory and Brownian motion.

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